ANALYTIC GEOMETRY AND CONIC SECTIONS

Precalculus
Chapter 7

- This Slideshow was developed to accompany the textbook
  - *Precalculus*
  - *By Richard Wright*
  - *https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html*
- Some examples and diagrams are taken from the textbook.

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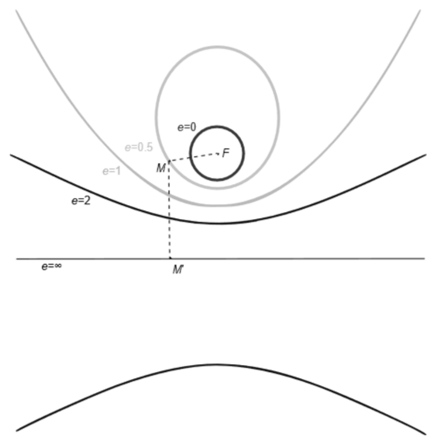
7-01 LINES

In this section, you will:

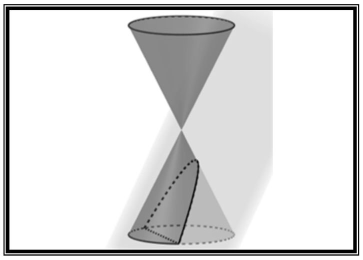
- Find the inclination of a line.
- Calculate the angle between two lines.
- Find the distance between a point and a line.

7-01 LINES

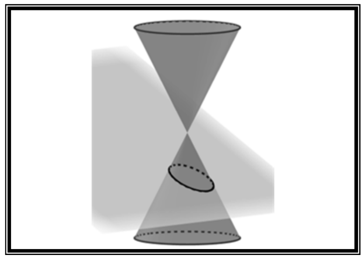
- Conic sections
  - Intersections of a plane with a double cone



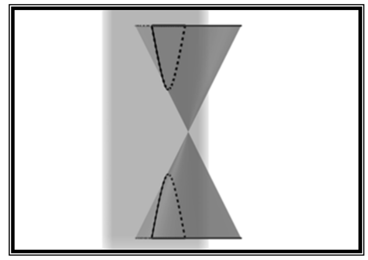
7-01 LINES



Parabola

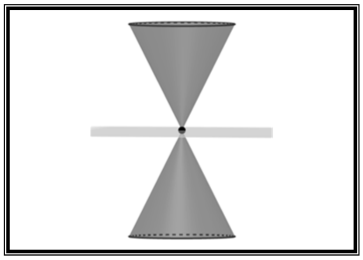


Ellipse



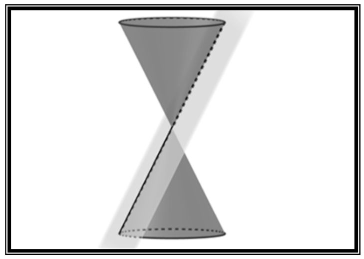
Hyperbola

7-01 LINES



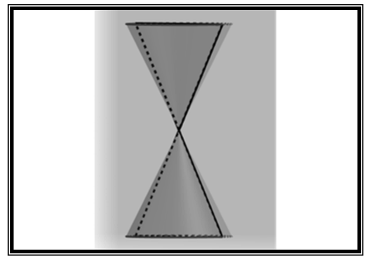
Point

Degenerate



Single Line

Degenerate



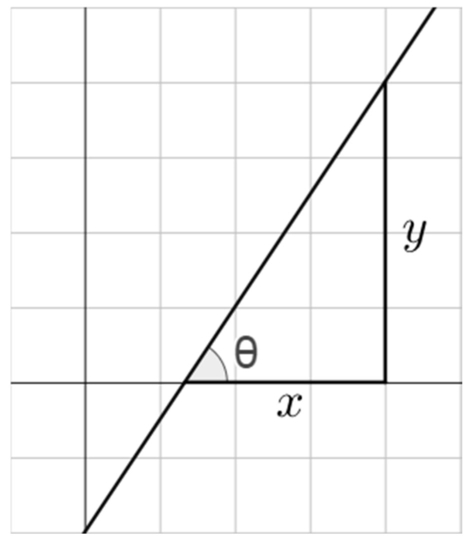
Intersecting Lines

Degenerate

The point and lines are called **degenerate conic sections** because they do not produce curves.

7-01 LINES

- Lines
  - $Ax + By + C = 0$ or $y = mx + b$
- Inclination
  - Describes steepness of line
  - Angle it makes with positive x-axis
  - $\tan\theta = \frac{y}{x} = slope$
  - $\tan\theta = m$
    - Where $0° < \theta < 180°$
    - If $\theta < 0$, add 180°



m = slope

7-01 LINES

- Find the inclination of $4x - 2y + 5 = 0$.

First, find the slope by rewriting the equation in slope-intercept form.

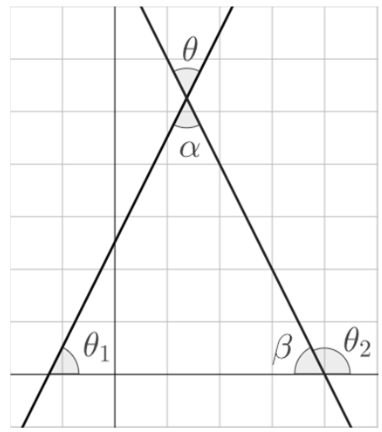
$$y = 2x + \frac{5}{2}$$

The slope is 2.
Find the inclination.

$$\tan\theta = 2$$
$$\theta = \tan^{-1} 2$$
$$\theta \approx 63.4°$$

7-01 LINES

- Angle between Two Lines
  - $\beta + \theta_2 = 180°$ (linear pair)
    - $\beta = 180° - \theta_2$
  - $\theta_1 + \alpha + \beta = 180°$ (triangle sum)
    - $\alpha = 180° - \theta_1 - \beta$
  - $\alpha = 180° - \theta_1 - (180° - \theta_2)$
  - $\alpha = \theta_2 - \theta_1$
  - $\theta = \alpha$ (vertical angles)
  - $\theta = \theta_2 - \theta_1$



7-01 LINES

- Written as slopes
  - $\tan\theta = \tan(\theta_2 - \theta_1)$
  - $\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$
  - Because the tangents are slopes
  - $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$
    - Where $0° < \theta < 90°$

7-01 LINES

- Find the angle between $2x + y = 4$ and $x - y = 2$.

Find the slopes

$$2x + y = 4$$

$y = -2x + 4$ $m = -2$

$$x - y = 2$$

$y = x - 2$ $m = 1$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1\, m_2}\right|$$

$$\tan\theta = \left|\frac{1 - (-2)}{1 + (1)(-2)}\right|$$

$$\tan\theta = \left|\frac{3}{-1}\right|$$

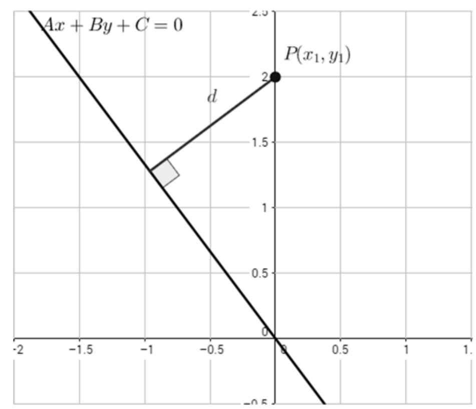
$$\tan\theta = 3$$

$$\theta = \tan^{-1} 3$$

$$\theta \approx 71.6°$$

7-01 LINES

- Distance from a Point to a Line
- This is derived in your book and online.
- Point (x_1, y_1) and Line $Ax + By + C = 0$
- $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$



7-01 LINES

- Find the distance from $(0, 2)$ to $4x + 3y = 0$.

The point is (x_1, y_1) so

$$x_1 = 0$$
$$y_1 = 2$$

The line is $Ax + By + C = 0$

$$A = 4, \mathrm{B} = 3, \mathrm{C} = 0$$
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$
$$d = \frac{|4(0) + 3(2) + 0|}{\sqrt{4^2 + 3^2}}$$
$$d = \frac{6}{5}$$

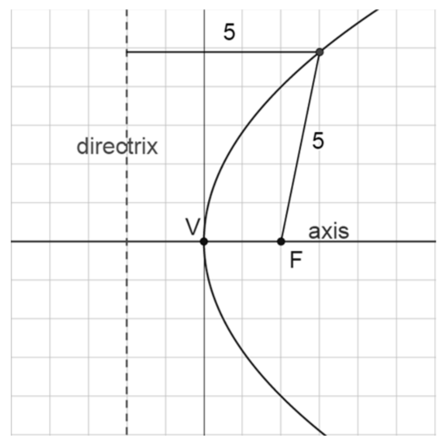
7-02 PARABOLAS

In this section, you will:

- Find the focus, vertex, and directrix of a parabola.
- Write the standard equation of a parabola.
- Graph a parabola.

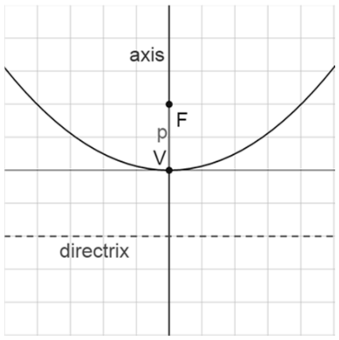
7-02 PARABOLAS

- Parabolas
  - set of all points in a plane that are equidistant from a fixed line, called the **directrix** and a fixed point, called the **focus**.
  - **Vertex**
    - max or min point
    - midpoint between the focus and directrix.
  - A**xis of symmetry**
    - line perpendicular to the directrix
    - goes through the focus and vertex.
  - Parabola bends around the focus and away from the directrix.



7-02 PARABOLAS

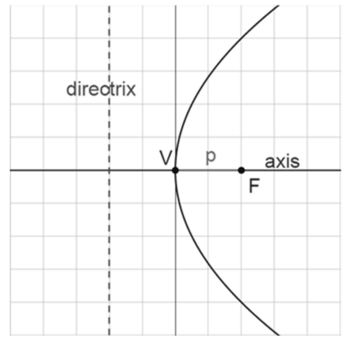
Vertical Parabola



p = directed (+, -) distance from vertex to focus
Vertex (h, k)
Focus $(h, p + k)$
Directrix $y = k - p$

$$(x - h)^2 = 4p(y - k)$$

Horizontal Parabola



p = directed (+, -) distance from vertex to focus
Vertex (h, k)
Focus $(p + h, k)$
Directrix $x = h - p$

$$(y - k)^2 = 4p(x - h)$$

7-02 PARABOLAS

- Find the vertex, focus, and directrix of the parabola given by $y = \frac{1}{2}x^2$.

Rearrange the equation to standard form

$$x^2 = 2y$$

Since it is x^2, it is a vertical parabola

$$(x-h)^2 = 4p(y-k)$$
$$h = 0, k = 0$$

$4p = 2$ so $p = \frac{1}{2}$

Vertex (h, k)

$$(0, 0)$$

Focus $(h, p + k)$

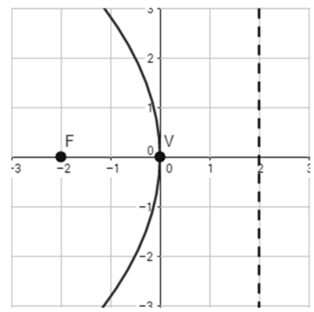
$$\left(0, \frac{1}{2} + 0\right) = \left(0, \frac{1}{2}\right)$$

Directrix

$$y = -\frac{1}{2}$$

7-02 PARABOLAS

- Find the standard form of the equations of a parabola with vertex at (0, 0) and focus (-2, 0).



The line through the points is horizontal
Vertex (*h*, *k*) = (0, 0)
Dist from vertex to focus = 2 left

p = -2

$$(y-k)^2 = 4p(x-h)$$
$$(y-0)^2 = 4(-2)(x-0)$$
$$y^2 = -8x$$

7-02 PARABOLAS

- Find the vertex, focus, and directrix of the parabola given by $x^2 - 2x - 16y - 31 = 0$.

x^2 so vertical parabola

Arrange the terms to fit standard form

$$x^2 - 2x = 16y + 31$$

Complete the square (add $\left(\frac{1}{2}b\right)^2$)

$$x^2 - 2x + \left(\frac{1}{2}(-2)\right)^2 = 16y + 31 + \left(\frac{1}{2}(-2)\right)^2$$

$$x^2 - 2x + 1 = 16y + 32$$

Factor

$$(x-1)^2 = 16(y+2)$$

Compare to $(x-h)^2 = 4p(y-k)$

$$h = 1, k = -2$$

$4p = 16$, so $p = 4$

Vertex (h, k)

$$(1, -2)$$

Focus $(h, p + k)$

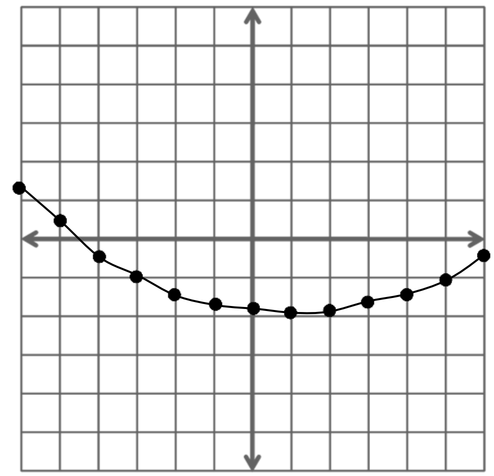
$$(1, 2)$$

Directrix $y = k - p$

$$y = -2 - 4 = -6$$

7-02 PARABOLAS

- Graph $(x-1)^2 = 16(y+2)$

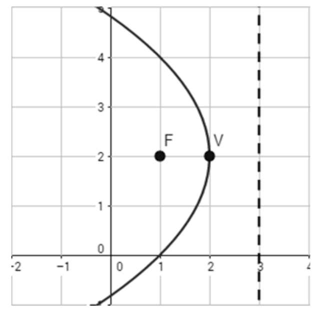


Solve for *y* and make a table of values

$$y = \frac{(x-1)^2}{16} - 2$$

7-02 PARABOLAS

- Write the standard form of the equation of the parabola with focus (1, 2) and directrix $x = 3$.



Directrix is vertical, so axis is horizontal
Graph the focus and directrix

Vertex is midway between (2, 2)

$$h = 2, k = 2$$

Distance from vertex to focus is 1 left

$$p = -1$$

$$(y - k)^2 = 4p(x - h)$$

$$(y - 2)^2 = 4(-1)(x - 2)$$

$$(y - 2)^2 = -4(x - 2)$$

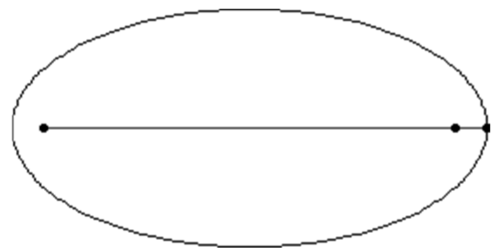
7-03 ELLIPSES AND CIRCLES

In this section, you will:

- Find the foci, vertices, and covertices of an ellipse.
- Write the standard equation of an ellipse.
- Graph an ellipse.

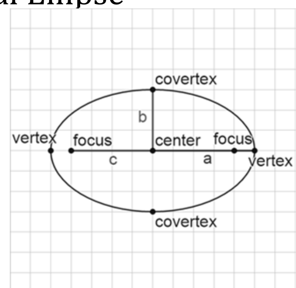
7-03 ELLIPSES AND CIRCLES

- **Ellipse**
  - Set of all points in a plane where the sum of the distances to two fixed points, **foci**, is constant.
  - M**ajor axis**
    - Longest segment across the ellipse
    - Connects the two **vertices**.
  - M**inor axis**
    - Shortest segment across the ellipse
    - Connects the two **covertices**.
- Circle
  - Special form of an ellipse where both foci are at the center.



7-03 ELLIPSES AND CIRCLES

Horizontal Ellipse



Center (h, k)
Horizontal Major Axis length = $2a$
Vertical Minor Axis length = $2b$

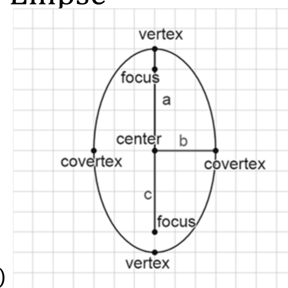
$$c^2 = a^2 - b^2$$

Vertices $(h \pm a, k)$
Covertices $(h, k \pm b)$
Foci $(h \pm c, k)$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

a = distance from center to vertex
b = distance from center to covertex
c = distance from center to focus

Vertical Ellipse



Center (h, k)
Vertical Major Axis length = $2a$
Horizontal Minor Axis length = $2b$

$$c^2 = a^2 - b^2$$

Vertices $(h, k \pm a)$
Covertices $(h \pm b, k)$
Foci $(h, k \pm c)$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

7-03 ELLIPSES AND CIRCLES

- Find the center, vertices, and foci of the ellipse $9x^2 + 4y^2 = 36$.

Put in standard form (÷ 36) to get 1

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Bigger denominator is a^2

$$\frac{y^2}{9} + \frac{x^2}{4} = 1$$

y is over big denominator so vertical

$$\frac{(y-k)^2}{a^2} + \frac{(b-h)^2}{b^2} = 1$$

$a^2 = 9$ so $a = 3$

$b^2 = 4$ so $b = 2$

$$c^2 = a^2 - b^2$$

$$c^2 = 9 - 4$$

$$c = \sqrt{5}$$

$$h = 0, k = 0$$

Center (h, k)

(0, 0)

Vertices $(h, k \pm a)$

$$(0, \pm 3)$$

Covertices $(h \pm b, k)$

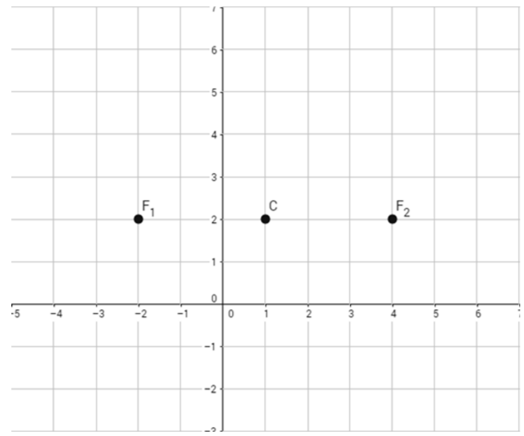
$$(\pm 2, 0)$$

Foci $(h, k \pm c)$

$$\left(0, \pm\sqrt{5}\right)$$

7-03 ELLIPSES AND CIRCLES

- Find the standard form of the ellipse centered at (1, 2) with major axis length 10 and foci (-2, 2) and (4, 2).



Graph the center and foci

Major axis is <u>horizontal</u>

Center (h, k) = (1, 2)

Major axis length 10 = 2*a*, so *a* = 5

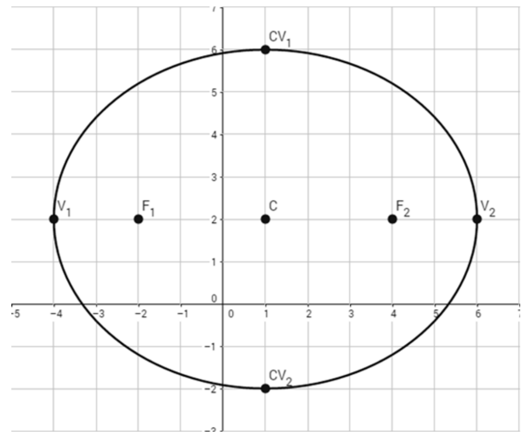
c is distance from center to foci, *c* = 3

$c^2 = a^2 - b^2$, so *b* = 4

$$\frac{(x-1)^2}{25} + \frac{(y-2)^2}{16} = 1$$

7-03 ELLIPSES AND CIRCLES

- Graph $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{16} = 1$



$a^2 = 25$, so $a = 5$
$b^2 = 16$, so $b = 4$
Vertices $(h \pm a, k)$
$(-4, 2)$ and $(6, 2)$
Covertices $(h, k \pm b)$
$(1, -2)$ and $(1, 6)$
Graph by plotting the vertices and covertices and drawing your best ellipse

7-03 ELLIPSES AND CIRCLES

- Sketch the graph of $25x^2 + 9y^2 - 200x + 36y + 211 = 0$

- Continues on next slide

Complete the square by moving the constant to the other side and factor *x*'s and *y*'s

$$25(x^2 - 8x) + 9(y^2 + 4y) = -211$$

Add $\left(\frac{1}{2}b\right)^2$ for both *x* and *y*

$$25\left(x^2 - 8x + \left(\frac{1}{2}(-8)\right)^2\right) + 9\left(y^2 + 4y + \left(\frac{1}{2}(4)\right)^2\right)$$

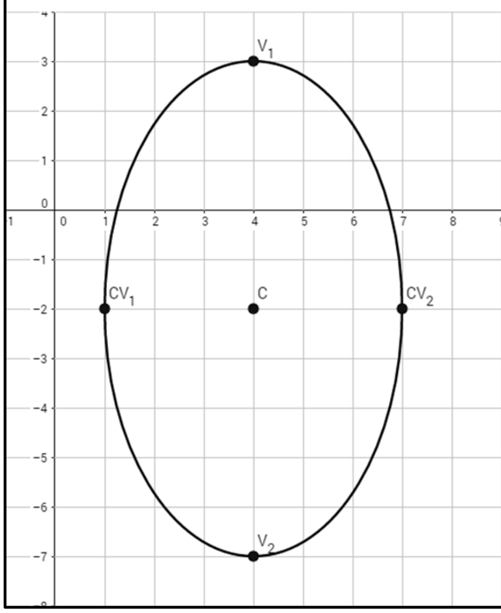
$$= -211 + 25\left(\frac{1}{2}(-8)\right)^2 + 9\left(\frac{1}{2}(4)\right)^2$$

$$25(x^2 - 8x + 16) + 9(y^2 + 4y + 4) = -211 + 400 + 36$$

Factor

$$25(x - 4)^2 + 9(y + 2)^2 = 225$$

7-03 ELLIPSES AND CIRCLES



$h = 4, k = -2$

$a^2 = 25$, so $a = 5$
$b^2 = 9$, so $b = 3$
Graph by plotting center (4, -2).
Vertices are up and down *a* = 5
$(4, 3)$ and $(4, -7)$
Covertices are left/right *b* = 3
$(1, -2)$ and $(7, -2)$

7-03 ELLIPSES AND CIRCLES

- Eccentricity
  - Measure of how circular an ellipse is
  - $e = \frac{c}{a}$ where $0 < e < 1$
  - If $e \approx 0$, then ellipse is almost a circle
  - If $e \approx 1$, then ellipse is almost a line

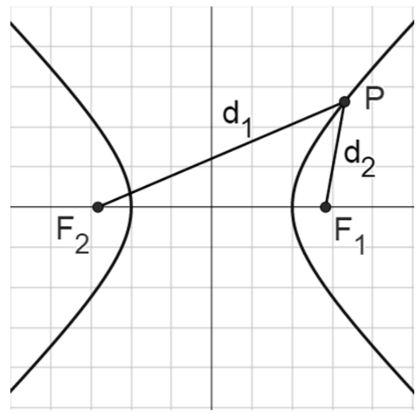
7-04 HYPERBOLAS

In this section, you will:

- Find the foci, vertices, covertices, and asymptotes of a hyperbola.
- Write the standard equation of a hyperbola.
- Graph a hyperbola.
- Classify conics based on the general equation.

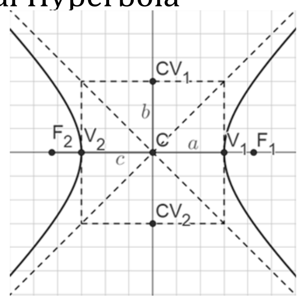
7-04 HYPERBOLAS

- **Hyperbolas**
  - Set of all points in a plane where the difference of the distances from two set points, foci, is constant.
  - $d_1 - d_2$ = constant.



7-04 HYPERBOLAS

Horizontal Hyperbola



Center (h, k)
Horizontal Transverse Axis length = $2a$
Vertical Conjugate Axis length = $2b$

$$c^2 = a^2 + b^2$$

Vertices $(h \pm a, k)$, Covertices $(h, k \pm b)$
Foci $(h \pm c, k)$

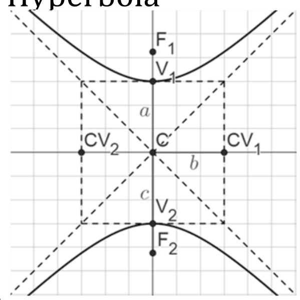
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Asymptotes $y = k \pm \frac{b}{a}(x-h)$

Eccentricity

- $e = \frac{c}{a}$
  Where $e > 1$
- Big e = linear branches

Vertical Hyperbola



Center (h, k)
Vertical Transvers Axis length = $2a$
Horizontal Conjugate Axis length = $2b$

$$c^2 = a^2 + b^2$$

Vertices $(h, k \pm a)$, Covertices $(h \pm b, k)$
Foci $(h, k \pm c)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Asymptotes $y = k \pm \frac{a}{b}(x-h)$

7-04 HYPERBOLAS

- Find the center, vertices, asymptotes, and foci of the hyperbola $4y^2 - 9x^2 = 36$.

Write in standard form by dividing by 36.

$$\frac{y^2}{9} - \frac{x^2}{4} = 1$$

Because *y* comes first, vertical hyperbola

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$h = 0, k = 0, a = 3, b = 2$$

$$c^2 = a^2 + b^2$$

$$c = \sqrt{13}$$

Center (h, k)

$$(0, 0)$$

Vertices $(h, k \pm a)$

$(0, -3)$ and $(0, 3)$

Asymptotes $y = k \pm \frac{a}{b}(x - h)$

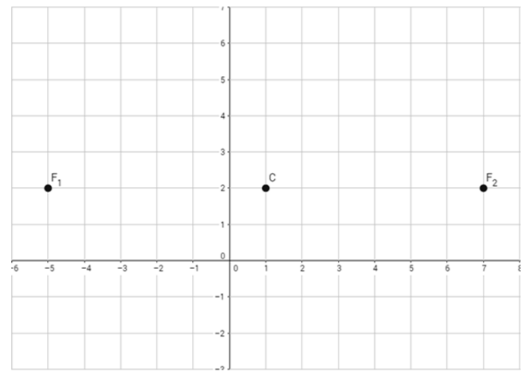
$$y = \pm\frac{3}{2}x$$

Foci $(h, k \pm c)$

$(0, -\sqrt{13})$ and $(0, \sqrt{13})$

7-04 HYPERBOLAS

- Find the standard form of the hyperbola centered at (1, 2) with transverse axis length 10 and foci (-5, 2) and (7, 2).



Graph the center and foci

Transverse axis is <u>horizontal</u>

Center (h, k) = (1, 2)

Transverse axis length 10 = 2*a*, so *a* = 5

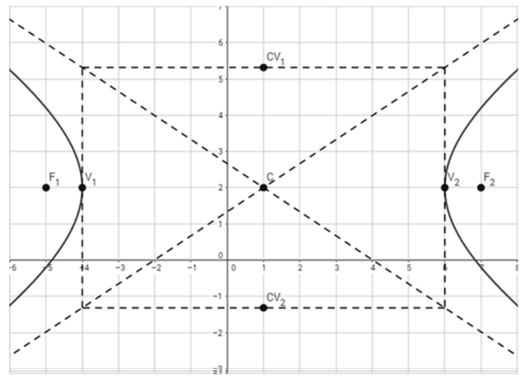
c is distance from center to foci, *c* = 6

$c^2 = a^2 + b^2$, so $b = \sqrt{11}$

$$\frac{(x-1)^2}{25} - \frac{(y-2)^2}{11} = 1$$

7-04 HYPERBOLAS

- Graph $\frac{(x-1)^2}{25} - \frac{(y-2)^2}{11} = 1$



$a^2 = 25$, so $a = 5$
$b^2 = 11$, so $b = \sqrt{11}$
Vertices $(h \pm a, k)$
    $(-4, 2)$ and $(6, 2)$
Covertices $(h, k \pm b)$
    $(1, -1.32)$ and $(1, 5.32)$
Graph by plotting the vertices and covertices
Drawing a rectangle
Draw diagonal lines through corners of rectangle
Sketch the hyperbola starting near asymptote, curve through vertex, end near other asymptote

7-04 HYPERBOLAS

- Sketch the graph of $4x^2 - 9y^2 - 24x - 72y - 72 = 0$

- Continues on next slide

Complete the square by moving the constant to the other side and factor *x*'s and *y*'s

$$4(x^2 - 6x) - 9(y^2 + 8y) = 72$$

Add $\left(\frac{1}{2}b\right)^2$ for both *x* and *y*

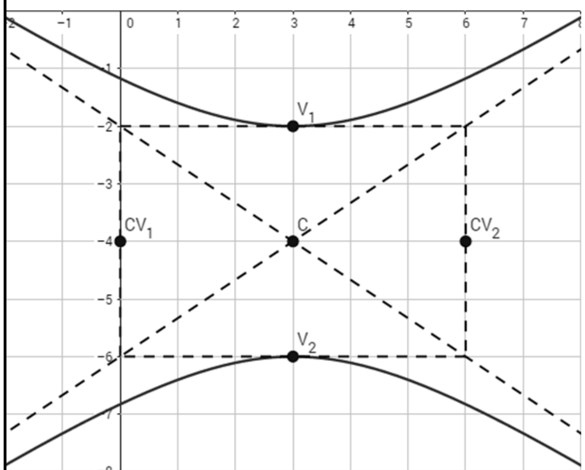
$$4\left(x^2 - 6x + \left(\frac{1}{2}(-6)\right)^2\right) - 9\left(y^2 + 8y + \left(\frac{1}{2}(8)\right)^2\right) = 72 + 4\left(\frac{1}{2}(-6)\right)^2 - 9\left(\frac{1}{2}(8)\right)^2$$

$$4(x^2 - 6x + 9) - 9(y^2 + 8y + 16) = 72 + 36 - 144$$

Factor

$$4(x - 3)^2 - 9(y + 4)^2 = -36$$

7-04 HYPERBOLAS



$h = 3, k = -4$

$a^2 = 4$, so $a = 2$
$b^2 = 9$, so $b = 3$
Graph by plotting center (3, -4).
Vertices are up and down *a* = 2
$(3, -2)$ and $(3, -6)$
Covertices are left/right *b* = 3
$(0, -4)$ and $(6, -4)$

7-04 HYPERBOLAS

- General form of conics
  - $Ax^2 + Cy^2 + Dx + Ey + F = 0$
  - Circle if $A = C$
  - Parabola if $AC = 0$ (so $A = 0$ or $C = 0$)
  - Ellipse if $AC > 0$
  - Hyperbola if $AC < 0$

- Classify the conics
- $4x^2 + 5y^2 - 9x + 8y = 0$
- $2x^2 - 5x + 7y - 8 = 0$
- $7x^2 + 7y^2 - 9x + 8y - 16 = 0$
- $4x^2 - 5y^2 - x + 8y + 1 = 0$

$AC = 4(5) = 20$ Ellipse

$AC = 2(0) = 0$ Parabola

$A = C = 7$ Circle

$AC = 4(-5) = -20$ Hyperbola

7-05 ROTATED CONICS

In this section, you will:

- Write rotated conics equations in standard form.
- Graph rotated conics.
- Classify conics by their equation.

7-05 ROTATED CONICS

- Nonrotated conics form $Ax^2 + Cy^2 + Dx + Ey + F = 0$.
  - Either horizontal or vertical.
- Rotated conics form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
  - Not horizontal or vertical
- *Bxy* term prevents completing the square to write the conics in standard form.
- To graph or write them in standard form, the *Bxy* term needs to be eliminated.
- Then write the equation in the form $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$ by rotating the coordinate axes counterclockwise through the angle θ, where
- $\cot 2\theta = \frac{A-C}{B}$
- Where $0 < 2\theta < 180°$ and $0 < \theta < 90°$

7-05 ROTATED CONICS

- **Classify Rotated Conics**
- If the conic is in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, then
  - If $B^2 - 4AC < 0 \rightarrow$ ellipse or circle
  - If $B^2 - 4AC = 0 \rightarrow$ parabola
  - If $B^2 - 4AC > 0 \rightarrow$ hyperbola

7-05 ROTATED CONICS

- **Write Rotated Conics in Standard Form**
- Given a conic written as $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

1. Find the angle of rotation using

$$\cot 2\theta = \frac{A - C}{B}$$

where $0 < \theta < \frac{\pi}{2}$

2. Find sin θ and cos θ.

- If θ is a special angle, evaluate sin θ and cos θ directly.
- If θ is not a special angle,
  a. Find cot 2θ.
  b. Reciprocal to find tan 2θ.
  c. Use $1 + \tan^2 u = \sec^2 u$ to find sec 2θ. (If tan 2θ < 0, then sec 2θ < 0.)
  d. Reciprocal to find cos 2θ.
  e. Use the half-angle formulas to find sin θ and cos θ.

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \text{ and } \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$$

7-05 ROTATED CONICS

3. Find the substitutions for *x* and *y* using

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

4. Make the substitutions and arrange the terms into standard form.

- **Graph a Rotated Conic**

1. Draw the rotated axes.
2. Using the rotated axes, sketch the conic.

7-05 ROTATED CONICS

- Write $xy = \frac{1}{2}$ in standard form

Classify the conic

$$B^2 - 4AC$$
$$1^2 - 4(0)(0) = 1 > 0$$

Hyperbola

Find the angle of rotation

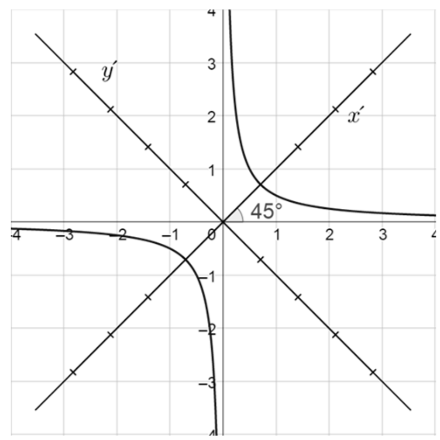
$$\cot 2\theta = \frac{A - C}{B} = \frac{0}{1} = 0$$
$$2\theta = \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$

Find the substitutions for *x* and *y*.

$$x = x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4}$$
$$y = x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4}$$

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'$$
$$y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

7-05 ROTATED CONICS



Substitute these into the original equation and simplify.

$$xy = \frac{1}{2}$$

$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) = \frac{1}{2}$$

$$\frac{1}{2}(x')^2 - \frac{1}{2}(x'y') + \frac{1}{2}(x'y') - \frac{1}{2}(y')^2 = \frac{1}{2}$$

$$(x')^2 - (y')^2 = 1$$

7-05 ROTATED CONICS

- Sketch the graph of
  $x^2 + \sqrt{3}xy + 2y^2 - 2 = 0.$

Classify the conic

$$B^2 - 4AC$$
$$\left(\sqrt{3}\right)^2 - 4(1)(2) = -5 < 0$$

Ellipse

Find the angle of the rotation.

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$
$$2\theta = \frac{2\pi}{3}$$

$\theta = \frac{\pi}{3}$

Find the substitutions for *x* and *y*.

$$x = x' \cos\frac{\pi}{3} - y' \sin\frac{\pi}{3}$$
$$y = x' \sin\frac{\pi}{3} + y' \cos\frac{\pi}{3}$$

$$x = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y'$$
$$y = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'$$

7-05 ROTATED CONICS

Substitute these into the original equation and simplify.

$$\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right)^2 + \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right)\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right) + 2\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right)^2 - 2 = 0$$

$$\frac{1}{4}(x')^2 - \frac{\sqrt{3}}{2}x'y' + \frac{3}{4}(y')^2 + \sqrt{3}\left(\frac{\sqrt{3}}{4}(x')^2 + \frac{1}{4}x'y' - \frac{3}{4}x'y' - \frac{\sqrt{3}}{4}(y')^2\right)$$
$$+ 2\left(\frac{3}{4}(x')^2 + \frac{\sqrt{3}}{2}x'y' + \frac{1}{4}(y')^2\right) - 2 = 0$$

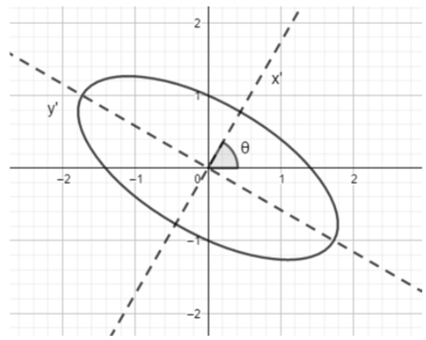
$$\frac{1}{4}(x')^2 - \frac{\sqrt{3}}{2}x'y' + \frac{3}{4}(y')^2 + \frac{3}{4}(x')^2 + \frac{\sqrt{3}}{4}x'y' - \frac{3\sqrt{3}}{4}x'y' - \frac{3}{4}(y')^2 + \frac{3}{2}(x')^2 + \sqrt{3}x'y'$$
$$+ \frac{1}{2}(y')^2 - 2 = 0$$

$$\frac{5}{2}(x')^2 + \frac{1}{2}(y')^2 = 2$$

Divide by to write to make the equation equal 1.

$$\frac{(x')^2}{4/5} + \frac{(y')^2}{4} = 1$$

7-05 ROTATED CONICS



$$\frac{(x')^2}{4/5} + \frac{(y')^2}{4} = 1$$

Vertical ellipse with $a = 2$ and $b = \frac{2\sqrt{5}}{5}$ and center (0, 0).

Draw the rotated axis, then move $a = 2$ along the rotated *y*-axis and $b = \frac{2\sqrt{5}}{5}$ along the rotated *x*-axis.

Connect the points with a nice ellipse.

7-05 ROTATED CONICS

Sketch the graph of
$3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0.$

Classify the conic

$$B^2 - 4AC$$
$$\left(2\sqrt{3}\right)^2 - 4(3)(1) = 0$$

Parabola

Find the angle of rotation.

$$\cot 2\theta = \frac{A-C}{B} = \frac{3-1}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$2\theta = \frac{\pi}{3}$$

$\theta = \frac{\pi}{6}$

Find the substitutions for *x* and *y*.

$$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6}$$
$$y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6}$$

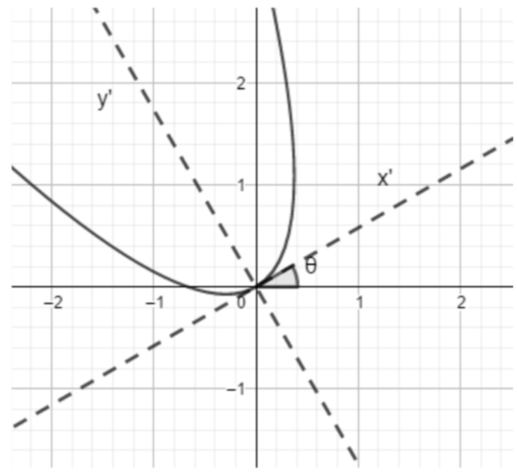
$$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$$
$$y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

7-05 ROTATED CONICS

- Substitute these into the original equation and simplify.

$$3\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 + 2\sqrt{3}\left(\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)\right) + \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2$$
$$+ 2\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right) - 2\sqrt{3}\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) = 0$$
$$3\left(\frac{3}{4}(x')^2 - \frac{\sqrt{3}}{2}x'y' + \frac{1}{4}(y')^2\right) + 2\sqrt{3}\left(\frac{\sqrt{3}}{4}(x')^2 + \frac{3}{4}x'y' - \frac{1}{4}x'y' - \frac{\sqrt{3}}{4}(y')^2\right)$$
$$+ \left(\frac{1}{4}(x')^2 + \frac{\sqrt{3}}{2}x'y' + \frac{3}{4}(y')^2\right) + \left(\sqrt{3}x' - y'\right) - \left(\sqrt{3}x' - 3y'\right) = 0$$
$$\frac{9}{4}(x')^2 - \frac{3\sqrt{3}}{2}x'y' + \frac{3}{4}(y')^2 + \frac{3}{2}(x')^2 + \frac{3\sqrt{3}}{2}x'y' - \frac{\sqrt{3}}{2}x'y' - \frac{3}{2}(y')^2 + \frac{1}{4}(x')^2 + \frac{\sqrt{3}}{2}x'y'$$
$$+ \frac{3}{4}(y')^2 + \sqrt{3}x' - y' - \sqrt{3}x' - 3y' = 0$$
$$4(x')^2 - 4y' = 0$$
$$y' = (x')^2$$

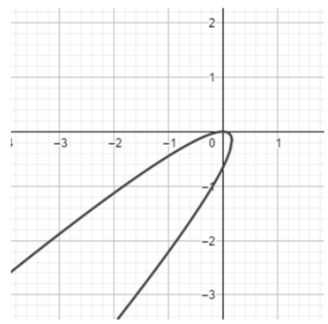
7-05 ROTATED CONICS



$y' = (x')^2$
This is a vertical parabola. Graph it by drawing the rotated axes and plotting points.

7-05 ROTATED CONICS

- Classify the graph, use the quadratic formula to solve for y, and use a graphing utility to graph the equation.
  $$3x^2 - 6xy + 3y^2 + 2y = 0$$



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Classify the graph using the discriminant.

$B^2 - 4AC = (-6)2 - 4(3)(3) = 0$, so it is a parabola.

To solve for *y*, rearrange terms in powers of *y* and factor.

$$3y^2 + (2y - 6xy) + 3x^2 = 0$$
$$3y^2 + (2 - 6x)y + 3x^2 = 0$$

Now fill in the quadratic formula: $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$y = \frac{-(2 - 6x) \pm \sqrt{(2 - 6x)^2 - 4(3)(3x^2)}}{2(3)}$$

Because of the ± sign, you will have to input two equations, one with + and one with -, to make the graph.

7-06 PARAMETRIC EQUATIONS

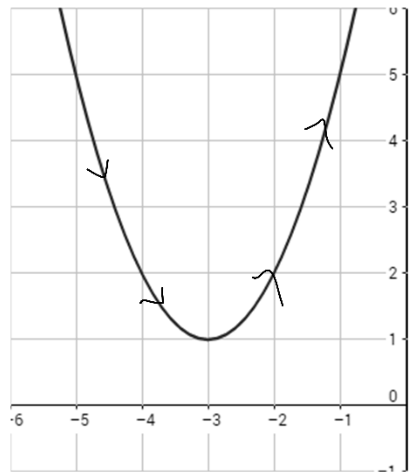
In this section, you will:

- Graph parametric equations.
- Write parametric equations.
- Eliminate the parameter.

7-06 PARAMETRIC EQUATIONS

- Parametric Equations
  - Separate equations for *x* and *y*
  - *x* and *y* are functions of a third variable called a parameter
- Graph $\begin{cases} x = t - 3 \\ y = t^2 + 1 \end{cases}$
  - Make a table

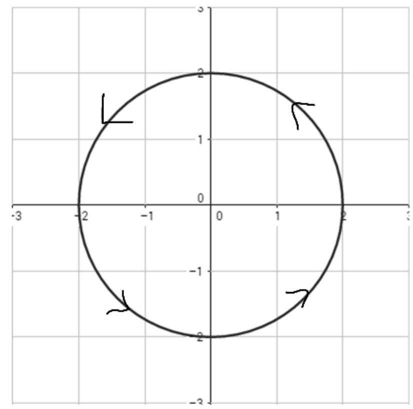
t	*x*	*y*
-2	-5	5
-1	-4	2
0	-3	1
1	-2	2
2	-1	5
3	0	10



7-06 PARAMETRIC EQUATIONS

- Graph $\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$ for $0 \le \theta \le 2\pi$

t	*x*	*y*
0	2	0
$\pi/4$	$\sqrt{2}$	$\sqrt{2}$
$\pi/2$	0	2
$3\pi/4$	$-\sqrt{2}$	$\sqrt{2}$
π	-2	0
$5\pi/4$	$-\sqrt{2}$	$-\sqrt{2}$
$3\pi/2$	0	-2
$7\pi/4$	$\sqrt{2}$	$-\sqrt{2}$



7-06 PARAMETRIC EQUATIONS

- Eliminating the Parameter
  - Solve one equation for parameter
  - Substitute it into the other equation
- Eliminate the parameter of $\begin{cases} x = \frac{1}{\sqrt{t}} \\ y = 2t^2 \end{cases}$

Solve one eq for t

$$x = \frac{1}{\sqrt{t}}$$
$$x\sqrt{t} = 1$$
$$t = \frac{1}{x^2}$$

Substitute into the other

$$y = 2(t^2)$$
$$y = 2\left(\frac{1}{x^2}\right)^2$$
$$y = \frac{2}{x^4}$$

7-06 PARAMETRIC EQUATIONS

- Eliminate the parameter in $\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$

Direct substitution is difficult so use identities

Remember $\sin^2\theta + \cos^2\theta = 1$

Square both equations

$$x^2 = 4\cos^2\theta$$
$$y^2 = 4\sin^2\theta$$

Add the new equations

$$x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta$$
$$x^2 + y^2 = 4(\cos^2\theta + \sin^2\theta)$$
$$x^2 + y^2 = 4$$

7-06 PARAMETRIC EQUATIONS

- Finding parametric equations
  - Choose something convenient to equal t

- Find parametric equations for
$$y = 4x - 3$$

Simple choice ($x = t$)

$$\begin{cases} x = t \\ y = 4t - 3 \end{cases}$$

Or more interesting ($t = 4x$)

$$\begin{cases} x = \dfrac{t}{4} \\ y = t - 3 \end{cases}$$

7-06 PARAMETRIC EQUATIONS

- Find parametric equations for conics.

- **Parabola**

- Horizontal: $\begin{cases} x = pt^2 + h \\ y = 2pt + k \end{cases}$

- Vertical: $\begin{cases} x = 2pt + h \\ y = pt^2 + k \end{cases}$

- **Ellipse**

- Horizontal: $\begin{cases} x = h + a\cos t \\ y = k + b\sin t \end{cases}$

- Vertical: $\begin{cases} x = h + b\sin t \\ y = k + a\cos t \end{cases}$

- **Hyperbola**

- Horizontal: $\begin{cases} x = h + a\sec t \\ y = k + b\tan t \end{cases}$

- Vertical: $\begin{cases} x = h + b\tan t \\ y = k + a\sec t \end{cases}$

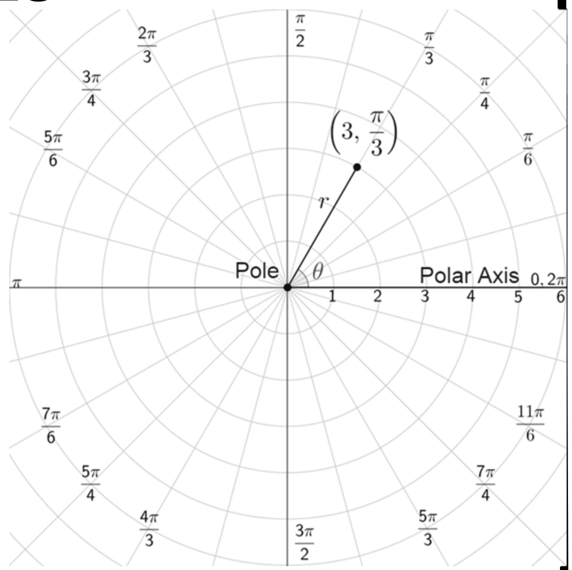
7-07 POLAR COORDINATES

In this section, you will:

- Graph polar coordinates
- Represent the same point multiple ways
- Convert between polar and rectangular coordinates

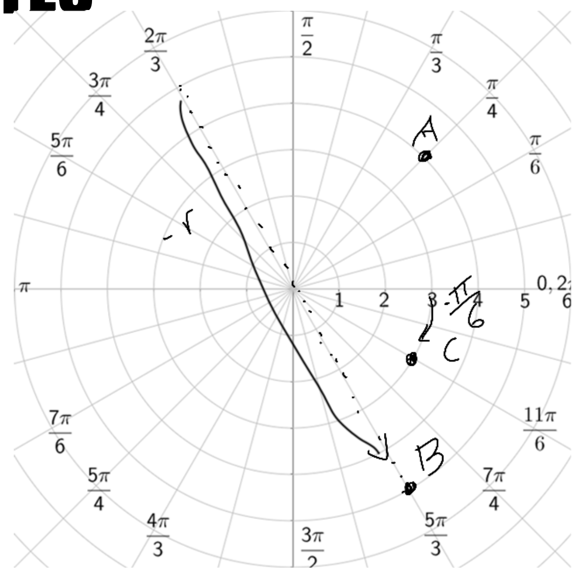
7-07 POLAR COORDINATES

- Why use rectangular coordinates to graph circles?
- Use circles to graph circles
- Polar coordinates
  - (r, θ)
  - r = distance from pole
  - θ = angle counterclockwise from polar axis



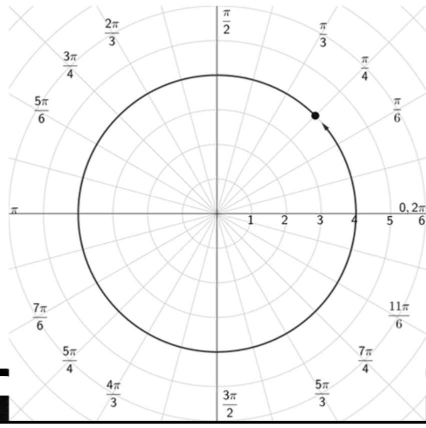
7-07 POLAR COORDINATES

- Graph
  - $A\left(4, \frac{\pi}{4}\right)$
  - $B\left(-5, \frac{2\pi}{3}\right)$
    - Negative r means go opposite
  - $C\left(3, -\frac{\pi}{6}\right)$
    - $= \left(3, \frac{11\pi}{6}\right)$

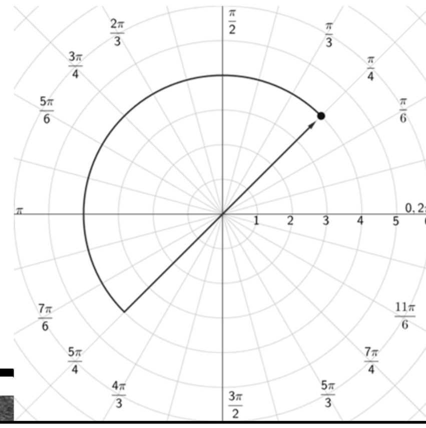


7-07 POLAR COORDINATES

- Multiple ways to represent same point
  - $(r, \theta) = (r, \theta \pm 2\pi n)$
    - Add full circles



- $(r, \theta) = (-r, \theta \pm (2n + 1)\pi)$
  - Opposite side of circle and add ½ circle



7-07 POLAR COORDINATES

- Find 2 other ways to write $\left(3, \frac{\pi}{4}\right)$.

Add a circle

$$\left(3, \frac{\pi}{4} + 2\pi\right)$$

$$\left(3, \frac{9\pi}{4}\right)$$

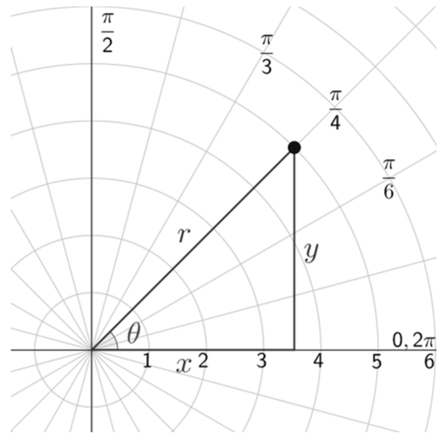
Move to opposite side and add ½ circle

$$\left(-3, \frac{\pi}{4} + \pi\right)$$

$$\left(-3, \frac{5\pi}{4}\right)$$

7-07 POLAR COORDINATES

- Convert between polar and rectangular
  - Polar → Rectangular
    - $x = r\cos\theta$
    - $y = r\sin\theta$
  - Rectangular → Polar
    - $r = \sqrt{x^2 + y^2}$
    - $\tan\theta = \frac{y}{x}$



7-07 POLAR COORDINATES

- Convert $\left(4, \frac{\pi}{6}\right)$ to rectangular
- Convert $(-1, 0)$ to polar

$$x = r\cos\theta$$
$$x = 4\cos\frac{\pi}{6}$$
$$x = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$
$$y = r\sin\theta$$
$$y = 4\sin\frac{\pi}{6}$$
$$y = 4\left(\frac{1}{2}\right) = 2$$
$$\left(2\sqrt{3}, 2\right)$$

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-1)^2 + (0)^2}$$
$$r = 1$$
$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{0}{-1}$$
$$\theta = \pi$$
$$(1, \pi)$$

7-07 POLAR COORDINATES

- Convert Equations
  - Convert $r = 1$
- Convert $\theta = \frac{\pi}{4}$

Substitute $r = \sqrt{x^2 + y^2}$

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1$$

Circle with $r = 1$

Use $\tan\theta = \frac{y}{x}$

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1$$

$$y = x$$

Line

7-07 POLAR COORDINATES

- Convert $r = \csc\theta$

Rewrite $r = \frac{1}{\sin\theta}$

$$r\sin\theta = 1$$

Use $y = r\sin\theta$

$$y = 1$$

Horizontal line

7-08 GRAPHS OF POLAR EQUATIONS

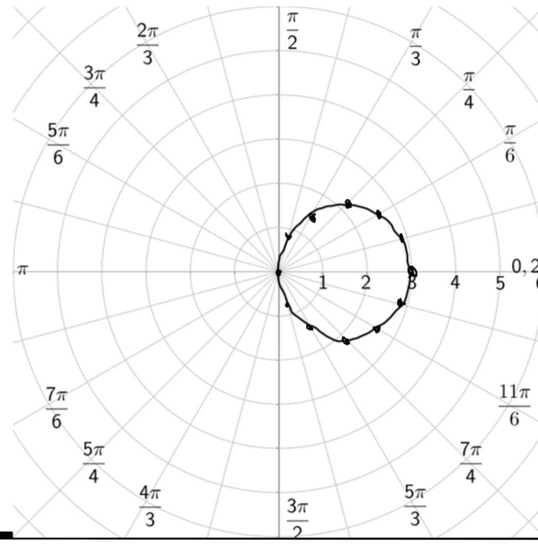
In this section, you will:

- Graph polar equations
- Identify symmetry in polar equations
- Find maximums and zeros of polar equations

7-08 GRAPHS OF POLAR EQUATIONS

- To graph polar equations using a table
  - Pick θ and calculate r
- Graph $r = 3\cos\theta$

r	3	2.9	2.6	2.1	1.5	0.8	0
θ	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
r	-0.8	-1.5	-2.1	-2.6	-2.9	-3	
θ	$7\pi/12$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$11\pi/12$	π	



7-08 GRAPHS OF POLAR EQUATIONS

- Symmetry Tests (make the replacement and to simplify to original equation)
  - Line $\theta = \frac{\pi}{2}$
    - Replace (r, θ) with $(r, \pi - \theta)$ or $(-r, -\theta)$
  - Polar Axis
    - Replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$
  - Pole
    - Replace (r, θ) with $(r, \pi + \theta)$ or $(-r, \theta)$
- Quick tests
  - If it is a function of $\cos\theta$, then polar axis symmetry
  - If it is a function of $\sin\theta$, then line $\theta = \frac{\pi}{2}$ symmetry

7-08 GRAPHS OF POLAR EQUATIONS

- Find the symmetry of $\theta = \frac{\pi}{4}$
- Line $\theta = \frac{\pi}{2}$:
  - Replace (r, θ) with $(-r, -\theta)$
- Polar axis
  - Replace (r, θ) with $(r, -\theta)$
- Pole
  - Replace (r, θ) with $(-r, \theta)$

$$-\theta = \frac{\pi}{4}$$
$$\theta = -\frac{\pi}{4}$$

This is different-no $\theta = \frac{\pi}{2}$ symmetry

$$-\theta = \frac{\pi}{4}$$
$$\theta = -\frac{\pi}{4}$$

No for polar axis symmetry

$$\theta = \frac{\pi}{4}$$

Same, YES for pole symmetry

7-08 GRAPHS OF POLAR EQUATIONS

- Find the symmetry of $r = 2(1 - \sin\theta)$

This is a function of $\sin\theta$, so quick test says

Line $\theta = \frac{\pi}{2}$ symmetry

7-08 GRAPHS OF POLAR EQUATIONS

- **Maximums and Zeros of Polar Equations**
- Maximums occurs when $|r|$ is largest.
  - Find angles where the trigonometric function is at its maximum.
- Zeros occur when $r = 0$.
  - Find angles where the trigonometric function is 0.

7-08 GRAPHS OF POLAR EQUATIONS

- Find the zeros and maximum r values of $r = 5\cos 2\theta$

Zeros

$$0 = 5\cos 2\theta$$
$$0 = \cos 2\theta$$
$$2\theta = \frac{\pi}{2} + n\pi$$
$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Maximums

Cos is *x* on unit circle, so max is when $2\theta = 0 + n\pi$

$$\theta = 0 + \frac{n\pi}{2}$$
$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

7-09 POLAR GRAPHS OF CONICS

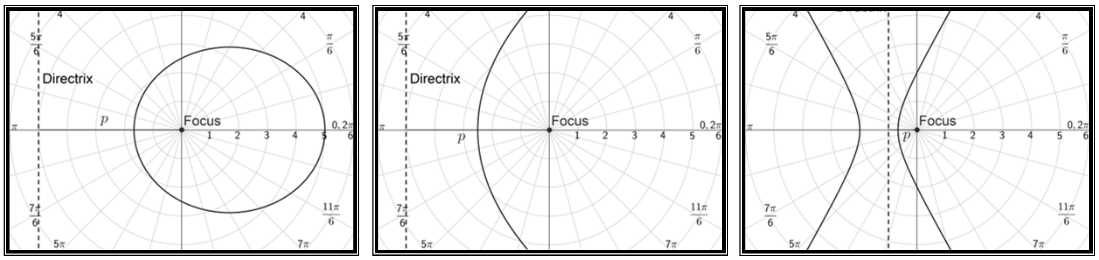
In this section, you will:

- Identify the type of a conic from its polar equation
- Find the polar equation of a conic

7-09 POLAR GRAPHS OF CONICS

- **Alternative Definition of a Conic Section**
- Locus of a point in the plane that moves so its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix).
- The ratio is the eccentricity (e).
  - $e < 1$ ellipse
  - $e = 1$ parabola
  - $e > 1$ hyperbola

7-09 POLAR GRAPHS OF CONICS



e < 1 Ellipse

e = 1 Parabola

e > 1 Hyperbola

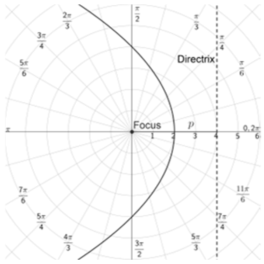
p = distance from focus to directrix

One focus is (0, 0)

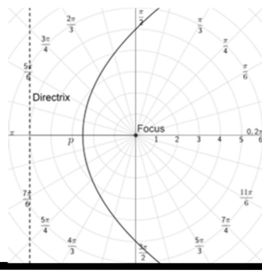
The conic bends around the focus and away from directrix

7-09 POLAR GRAPHS OF CONICS

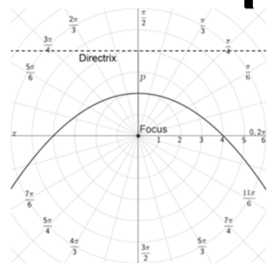
- Vertical Directrix
  - Right of pole
    - $r = \frac{ep}{1+e\cos\theta}$



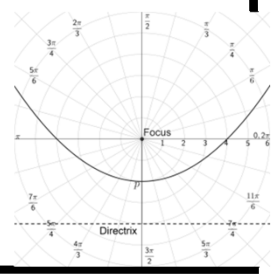
  - Left of pole
    - $r = \frac{ep}{1-e\cos\theta}$



- Horizontal Directrix
  - Above pole
    - $r = \frac{ep}{1+e\sin\theta}$

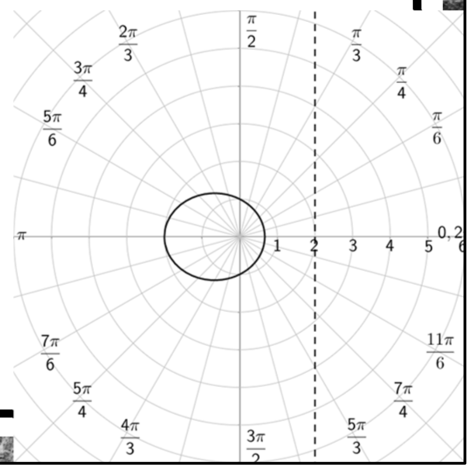


  - Below pole
    - $r = \frac{ep}{1-e\sin\theta}$



7-09 POLAR GRAPHS OF CONICS

- Identify the type of conic $r = \frac{2}{2+\cos\theta}$



Want it in the form $r = \frac{ep}{1+e\cos\theta}$, so multiply top and bottom by ½ to get the 1

$$r = \frac{\frac{1}{2}(2)}{\frac{1}{2}(2+\cos\theta)}$$

$$r = \frac{1}{1+\frac{1}{2}\cos\theta}$$

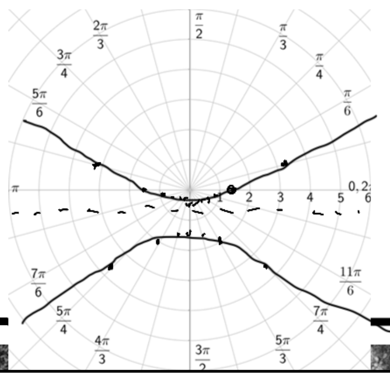
This is vertical directrix to right of pole

$e = \frac{1}{2} < 1$ so Ellipse

7-09 POLAR GRAPHS OF CONICS

- Identify type of conic and graph

$$r = \frac{3}{2 - 4\sin\theta}$$



Want a 1 at beginning of denominator, so multiply top and bottom by ½

$$r = \frac{\frac{1}{2}(3)}{\frac{1}{2}(2 - 4\sin\theta)}$$

$$r = \frac{\frac{3}{2}}{1 - 2\sin\theta}$$

This is like $r = \frac{ep}{1 - e\sin\theta}$

Horizontal directrix below pole

$e = 2 > 1$ Hyperbola

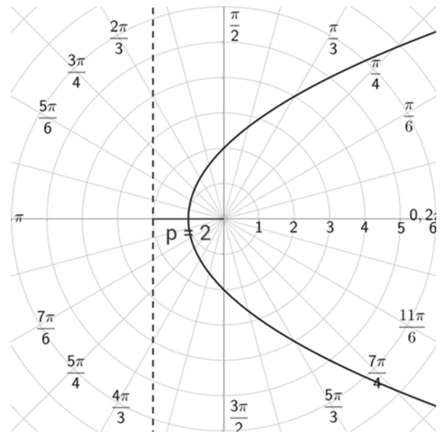
$$ep = \frac{3}{2}$$

$2p = \frac{3}{2} \rightarrow p = \frac{3}{4}$

Graph with a table

7-09 POLAR GRAPHS OF CONICS

- Find the polar equation of the parabola whose focus is the pole and directrix is the line $x = -2$.



Graph the directrix and count to find *p*

$p = 2$

Parabola so $e = 1$

Vertical directrix to left of pole

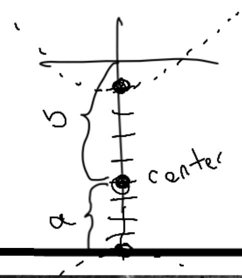
$$r = \frac{ep}{1 - e\cos\theta}$$

$$r = \frac{2(1)}{1 - (1)\cos\theta}$$

$$r = \frac{2}{1 - \cos\theta}$$

7-09 POLAR GRAPHS OF CONICS

- Find the polar equation of the hyperbola with focus at pole and vertices $\left(1, \frac{3\pi}{2}\right)$ and $\left(-9, \frac{\pi}{2}\right)$.



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Graph the vertices.

The center is midpoint between vertices

a = center to vertex = 4

c = center to focus = 5

$$e = \frac{c}{a} = \frac{5}{4}$$

Horizontal directrix below pole

$$r = \frac{ep}{1 - e\sin\theta}$$

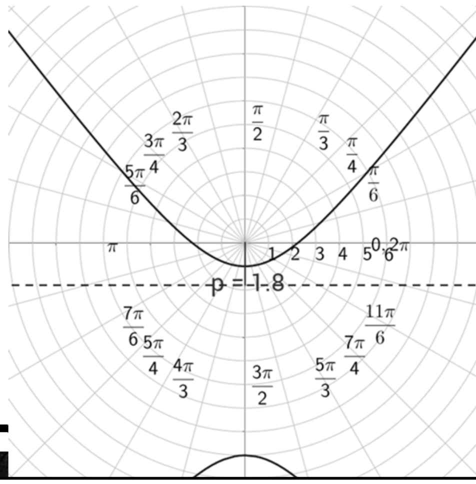
$$r = \frac{\frac{5}{4}p}{1 - \frac{5}{4}\sin\theta}$$

Multiply top and bottom by 4 to remove fractions

$$r = \frac{5p}{4 - 5\sin\theta}$$

7-09 POLAR GRAPHS OF CONICS

- Plug in a point $\left(1, \frac{3\pi}{2}\right)$ to find p



- Write final equation

$$r = \frac{5p}{4 - 5\sin\theta}$$
$$1 = \frac{5p}{4 - 5\sin\frac{3\pi}{2}}$$
$$1 = \frac{5p}{4 - 5(-1)}$$
$$1 = \frac{5p}{9}$$
$$9 = 5p$$

$$r = \frac{5p}{4 - 5\sin\theta}$$
$$r = \frac{9}{4 - 5\sin\theta}$$